Interpreting Regression Coefficients
Step-by-Step Derivations and Explanations

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1 Level-Level Models

A level-level regression model is a model in which the dependent variable (Y) and the independent variables (the X's) have not been transformed in any way. That is: they are both in levels. These types of models can be expressed as follows.

(1) $Y = b_0 + b_1 X_1 + \cdots + b_k X_k + e$

The coefficient b_1 is interpreted as follows $\frac{\partial Y}{\partial X_1} = b_1$. The partial derivative of Y with respect to X_1, $\frac{\partial Y}{\partial X_1}$, can be interpreted as the rate of change in Y associated with a change in X_1 holding constant all other X's when there is more than one X variable in the model.

By changing the symbol ∂ for $\triangle$, we can also write this as:

$\frac{\triangle Y}{\triangle X_1} = b1 \implies$

$\triangle Y = b1 \cdot \triangle X_1$

In other words if we change X_1 by 1 unit (i.e. $\triangle X_1 = 1$, then we can expect Y to change by b_1 units. That is: we expect that $\triangle Y = b_1$).

So the interpretation of b_1 in a level-level regression is that a 1 unit change in X_1 is associated with a b_1 unit change in Y *holding constant all other variables in the model.*

2 Log-Log Models

A log-log model occurs when both the dependent variable (Y) and the right hand side variables are in log form. These models can be expressed as follows:

(2) $ln(Y) = b_0 + b_1 ln(X_1) + \cdots + b_k ln(X_k) + e$

One way to derive a meaningful interpretation of the coefficient b_1 is to exponentiate both the left hand and right hand side of the model as follows:

(3) $e^{ln(Y)} = e^{(b_0 + b_1 ln(X_1) + \cdots + b_k ln(X_k) + e)}$

By applying the rules of logarithms and exponents, equation (3) is equivalent to:

$Y = e^{(b_0 + b_1 ln(X_1) + \cdots + b_k ln(X_k) + e)}$

Now if we take the partial derivative of Y with respect toX_1 we wind up with:

$\frac{\partial Y}{\partial X_1} = b_1 \frac{\partial ln(X_1)}{\partial X_1} \cdot e^{(b_0+b_1 ln(X_1)+\cdots+b_k ln(X_k)+e)} = b_1 \frac{1}{X_1} \cdot e^{(b_0+b_1 ln(X_1)+\cdots+b_k ln(X_k)+e)}$

If we divide both sides by Y, we get:

(4) $\frac{\partial Y}{\partial X_1}\frac{1}{Y} = b_1 \frac{1}{X_1} \frac{e^{(b_0+b_1 ln(X_1)+\cdots+b_k ln(X_k)+e)}}{Y} = b_1 \frac{1}{X_1} \frac{e^{(b_0+b_1 ln(X_1)+\cdots+b_k ln(X_k)+e)}}{e^{(b_0+b_1 ln(X_1)+\cdots+b_k ln(X_k)+e)}} = b_1 \frac{1}{X_1}$

because $Y = e^{(b_0+b_1 ln(X_1)+\cdots+b_k ln(X_k)+e)}$

So exchanging the ∂ symbol for the $\triangle$ symbol, we can rewrite (4) as $\frac{\triangle Y}{\triangle X_1}\frac{1}{Y} = b_1 \frac{1}{X_1}$

and if we multiply both sides by $\triangle X_1$ then we wind up with:

(5) $\frac{\triangle Y}{Y} = b_1 \frac{\triangle X_1}{X_1}$

Multiplying both sides by 100 gives us the following formula:

$100 \cdot \frac{\triangle Y}{Y} = b_1 \cdot 100 \cdot \frac{\triangle X_1}{X_1} \iff \triangle\%Y = b_1 \triangle\%X_1$

So the interpretation in a log-log model is that a 1% change in X_1 is associated with a b_1 % change in Y *holding constant all other variables in the model*.

3 Level-Log Models

A level-log model occurs when the dependent variable (Y) is in level form and the right hand side variables are in log form. These models can be expressed as follows:

(6) $Y = b_0 + b_1 ln(X_1) + \cdots + b_k ln(X_k) + e$

Taking the derivative of Y with respect to X_1, we get:

$\frac{\partial Y}{\partial X_1} = b_1 \frac{\partial ln(X_1)}{\partial X_1} = b_1 \cdot \frac{1}{X_1}$

Substituting in the $\triangle$ symbol for the ∂ symbol, we wind up with:

$\frac{\triangle Y}{\triangle X_1} = b_1 \cdot \frac{1}{X_1}$

Multiplying both sides by $\triangle X_1$ gives us:

$\triangle Y = b_1 \cdot \frac{\triangle X_1}{X_1}$

Multiplying both sides by 100 gives us the following:

$100 \cdot \triangle Y = b_1 \cdot 100 \cdot \frac{\triangle X_1}{X_1} \iff 100 \cdot \triangle Y = b_1 \cdot \triangle\%X_1 \implies \triangle Y = \frac{b_1}{100} \cdot \triangle\%X_1$

Therefore, in a level-log model, the interpretation is that a 1% increase in X is associated with a $\frac{b_1}{100}$ unit change in Y *holding constant all other variables in the model.*

4 Log-Level Models

A log-level model occurs when the dependent variable (Y) is in log form and the right hand side variables are in level form. These models can be expressed as follows:

(7) $ln(Y) = b_0 + b_1 X_1 + \cdots + b_x X_k + e$

One way to derive a meaningful interpretation of the coefficient b_1 is to exponentiate both the left hand and right hand side of the models.

(8) $e^{ln(Y)} = e^{(b_0 + b_1 X_1 + \cdots + b_k X_k + e)}$

By applying the rules of logarithms and exponents, equation (8) is equivalent to:

$Y = e^{(b_0 + b_1 X_1 + \cdots + b_k X_k + e)}$

Now if we take the partial derivative of Y with respect to X_1 we wind up with:

$\frac{\partial Y}{\partial X_1} = b_1 \cdot e^{(b_0 + b_1 X_1 + \cdots + b_k X_k + e)}$

If we divide both sides by Y, we get:

(9) $\frac{\partial Y}{\partial X_1} \frac{1}{Y} = b_1 \frac{e^{(b_0 + b_1 X_1 + \cdots + b_k X_k + e)}}{Y} = b_1 \frac{e^{(b_0 + b_1 X_1 + \cdots + b_k X_k + e)}}{e^{(b_0 + b_1 X_1 + \cdots + b_k X_K + e)}} = b_1$

because $Y = e^{(b_0 + b_1 X_1 + \cdots + b_k X_k + e)}$

So we can rewrite (9) as $\frac{\triangle Y}{\triangle X_1} \frac{1}{Y} = b_1$

and if we multiply both sides by $\triangle X_1$ then we wind up with:

$\frac{\triangle Y}{Y} = b_1 \triangle X_1$

Multiplying both sides by 100 gives us the following expression:

$100 \cdot \frac{\triangle Y}{Y} = 100 \cdot b_1 \triangle X_1 \iff \triangle \% Y = 100 \cdot b_1 \triangle X_1$

So the interpretation is that a 1 unit change in X_1 is associated with a $100 \cdot b_1$ % change in Y *holding constant all other variables in the model.*